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Invariant hyperplanes and Darboux integrability of polynomial vector fields

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Abstract
This paper is composed of two parts. In the first part, we provide an upper bound for the number of invariant hyperplanes of the polynomial vector fields in n variables. This result generalizes those given in Artés *et al* (1998 *Pac. J. Math.* **184** 207–30) and Llibre and Rodríguez (2000 *Bull. Sci. Math.* **124** 599–619). The second part gives an extension of the Darboux theory of integrability to polynomial vector fields on algebraic varieties.

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1. Introduction and statement of the main results

Nonlinear ordinary differential equations appear in many branches of applied mathematics and physics, for instance the Lorenz system, the Lotka–Volterra system and so on. Here we consider only the polynomial differential systems.

This paper is separated into two parts. The first part is related to the maximum number of invariant hyperplanes. The second part provides an extension of the Darboux theory of integrability, which gives a link between the invariant algebraic hypersurfaces and the first integrals. Hence, these two parts are related closely.

We must mention that not only in mathematics, but also in physics, the research on invariant hyperplanes and invariant surfaces is important. For example, using invariant surfaces Giacomini and Neukirch [7] constructed families of two-dimensional surfaces transverse to the flow of the Lorenz system, such that each of the surfaces separates the phase space $\mathbb{R}^3$ and hence can be used to describe the location of the global attractor of the flow. In [2, 8, 11] the authors studied the integrals of motion for some famous three-dimensional non-Hamiltonian dynamical systems. In fact, every integral of motion can be obtained from an invariant surface or an invariant plane with a constant cofactor. So, searching for integrals of motion is equivalent to obtaining the mentioned invariant surfaces or invariant planes.

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The integrability of a dynamical system is also one of the important subjects considered by physicists. Because when a system has a first integral, we can restrict the motion of the orbits to a suitable surface. In really dynamical models, the motions of the objects are usually controlled on some given surface or variety, but not in the whole space. It is one of the motivations that we extend the Darboux integrability in the whole space to that on a variety.

This paper is organized as follows. In sections 1.1 and 1.2 we give some definitions and state our main results on invariant hyperplanes and the extension of Darboux integrability, respectively. In sections 2 and 3 we prove theorems 1 and 2, respectively. The proof of theorem 3 is given in section 4.

1.1. Invariant hyperplanes

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or other algebraically closed fields, and let $\mathbb{K}^n$ be the affine n space over $\mathbb{K}$ formed by all n-tuples of elements of $\mathbb{K}$. By definition, a *polynomial vector field* in $\mathbb{K}^n$ is of the form

$$\mathbf{X} = P_1(x_1, \ldots, x_n)\frac{\partial}{\partial x_1} + \cdots + P_n(x_1, \ldots, x_n)\frac{\partial}{\partial x_n} \tag{1}$$

where $P_1, \ldots, P_n \in \mathbb{K}[x_1, \ldots, x_n]$. As usual, $\mathbb{K}[x_1, \ldots, x_n]$ denotes the polynomial ring over $\mathbb{K}$ in the variables $x_1, \ldots, x_n$. We state that the vector field $\mathbf{X}$ has *degree* m if $m = \max\{\deg P_i, i = 1, \ldots, n\}$. More precisely, we state that $\mathbf{X}$ has *degree* $\mathbf{m} = (m_1, \ldots, m_n)$ if $m_i = \deg P_i$ for $i = 1, \ldots, n$.

An *invariant hyperplane* of the vector field $\mathbf{X}$ is a hyperplane $f(x_1, \ldots, x_n) = 0$ with $f \in \mathbb{K}[x_1, \ldots, x_n]$ a polynomial of degree 1 and $\mathbf{X}f = kf$ for some $k \in \mathbb{K}[x_1, \ldots, x_n]$. In particular, we call it an *invariant plane* if $n = 3$, or an *invariant line* if $n = 2$. In the definitions of the last three notions, if $\deg f > 1$, we call $f(x_1, \ldots, x_n) = 0$ (or simply f) an *invariant algebraic hypersurface* if $n > 3$, an *invariant algebraic surface* if $n = 3$, or an *invariant algebraic curve* if $n = 2$.

If the vector field $\mathbf{X}$ has finitely many invariant hyperplanes, we denote by $\alpha(n, \mathbf{m})$ (respectively $\alpha(n, m)$) the supremum of the number of hyperplanes invariant by the vector field $\mathbf{X}$ of degree $\mathbf{m} = (m_1, \ldots, m_n)$ (respectively m) in n variables.

For planar polynomial vector fields of degree m, Artés *et al* [1, proposition 4] proved that $\alpha(2, m) \leqslant 3m - 1$. Sokulski [19] and Zhang and Ye [23] obtained $\alpha(2, 3) = 8$ and $\alpha(2, 4) = 9$. Artés *et al* [1] obtained $\alpha(2, 5) = 14$. For $m > 5$, *what is the exact value of* $\alpha(2, m)$? It still remains an open problem.

For polynomial vector fields of degree $\mathbf{m} = (m_1, \ldots, m_n)$ in $\mathbb{K}^n$, Llibre and Rodríguez [13, proposition 6] proved that if $\mathbf{X}$ is a *regular polynomial vector field*, then

$$\alpha(n, \mathbf{m}) \leqslant \sum_{i=1}^{n} m_i + (m-1)\binom{n}{2} = \beta(n, \mathbf{m})$$

where $m = \max\{m_i, i = 1, \ldots, n\}$. Moreover, they showed that if $\mathbf{m} = (2, \ldots, 2)$ or $\mathbf{m} = (3, \ldots, 3)$ then $\alpha(n, \mathbf{m}) = \beta(n, \mathbf{m})$, and posed the following open problem: *determine the exact value of* $\alpha(n, \mathbf{m})$.

Our first result generalizes proposition 4 of [1] and proposition 6 of [13].

Theorem 1. *For any polynomial vector fields of degree* $\mathbf{m}$ *in* $\mathbb{K}^n$*, we have*

$$\alpha(n, \mathbf{m}) \leqslant \sum_{i=1}^{n} m_i + (m-1)\binom{n}{2} = \beta(n, \mathbf{m}).$$

This theorem shows that if $n = 2$, then $\alpha(2, m) \leqslant m_1 + m_2 + (m - 1) \leqslant 3m - 1$. It improves proposition 4 of [1]. The following example provides an application of the theorem to a dynamical model.

Example 1. The two-dimensional Lotka–Volterra system

$$\dot{x} = x(a_1 x + b_1 y + c_1) \qquad \dot{y} = y(a_2 x + b_2 y + c_2)$$

with a_i, b_i, c_i real constants, under the conditions $a_2 = b_1 = 0$, $c_1 = c_2$ and $a_1 b_2 \neq 0$, has the maximum number of, i.e. five, invariant lines: $f_1 = y + \frac{c_1}{b_2}$ with the cofactor $k_1 = b_2 y$, $f_2 = x + \frac{c_1}{a_1}$ with the cofactor $k_2 = a_1 x$, $f_3 = x - \frac{b_2}{a_1} y$ with the cofactor $k_3 = a_1 x + b_2 y + c_1$ and the two known ones $f_4 = x$ with the cofactor $k_4 = a_1 x + c_1$ and $f_5 = y$ with the cofactor $k_5 = b_2 y + c_1$, respectively.

Of course, under other suitable conditions we can get five invariant lines for the last system. In section 1.2 we will use these five invariant lines and the Darboux theory of integrability to construct a first integral for the two-dimensional Lotka–Volterra system.

The following theorem answers the open problem of [13] for $m = 1$.

Theorem 2. *For $m = \max\{m_i, i = 1, \ldots, n\} = 1$, the following statements hold:*

(a) The necessary condition for a linear vector field of degree $\mathbf{m} = (m_1, \ldots, m_n)$ to have finitely many invariant hyperplanes is $\mathbf{m} = (1, \ldots, 1)$ or $\mathbf{m} = (0, 1, \ldots, 1)$.
(b) There exist linear vector fields of degree $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_1 = \cdots = m_n = 1$, or $m_1 = 0$ and $m_2 = \cdots = m_n = 1$ such that the vector fields have exactly $\sum_{i=1}^{n} m_i = \beta(n, \mathbf{1})$ invariant hyperplanes.
(c) The set of linear vector fields with exactly $\sum_{i=1}^{n} m_i$ invariant hyperplanes is open and dense in the linear space formed by all linear vector fields.

We remark that the notion of open and dense is considered under the well-known coefficient topology on the space of linear vector fields.

1.2. The Darboux integrability of polynomial vector fields on algebraic varieties

In 1878, Darboux [5] provided a link between algebraic geometry and the search of first integrals, and showed how to construct the first integral of polynomial vector fields in $\mathbb{R}^2$ or $\mathbb{C}^2$ having sufficiently many invariant algebraic curves. The extensions of the Darboux theory of integrability to polynomial systems in $\mathbb{R}^n$ or $\mathbb{C}^n$ are due to Jouanolou [10] and Weil [20]. In [3, 4, 15] the authors developed the Darboux theory of integrability essentially in $\mathbb{R}^2$ or $\mathbb{C}^2$ considering not only the invariant algebraic curves but also the exponential factors, the independent singular points and the multiplicity of the invariant algebraic curves. Recently, in [9, 14, 16] there are extensions of the Darboux theory of integrability to polynomial systems on regular surfaces. In this paper, we extend the theory to polynomial vector fields on algebraic varieties.

Let $\mathbb{K}$ be a fixed algebraically closed field. We assume that $\mathcal{V} \subset \mathbb{K}^n$ is an affine algebraic variety. Then $I(\mathcal{V}) = \{f \in \mathbb{K}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in \mathcal{V}\}$ is a prime ideal (see Fulton [6, page 15]). Furthermore, we assume that $\mathcal{V}$ is smooth, i.e. there exist $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$ with $k < n$ such that $\mathcal{V} = V(f_1, \ldots, f_k) = \{p \in \mathbb{K}^n : f_i(p) = 0,$ for $i = 1, \ldots, k\}$ and rank $\mathcal{J}(f) = k$ on $\mathcal{V}$, where $\mathcal{J}(f)$ is the Jacobian matrix of $f = (f_1, \ldots, f_k)$.

Two polynomials f and g in $\mathbb{K}[x_1,\ldots,x_n]$ are said to be *related*, denoted by $f \sim g$, if $f-g \in I(\mathcal{V})$. By Hilbert's Nullstellensatz, if this is the case there exist $N \in \mathbb{N}$ and $A_1,\ldots,A_k \in \mathbb{K}[x_1,\ldots,x_n]$ such that

$$(f-g)^N = A_1 f_1 + \cdots + A_k f_k.$$

We denote by $\overline{f}$ the set of all polynomials related with f.

For any $g, h \in \overline{f}$, we state that $g < h$ if $\deg g < \deg h$. Thus, $\overline{f}$ forms a partial order set. It has a minimum element. In what follows, $\overline{f}$ always means that f is the minimum element in $\overline{f}$. We call $\overline{f}$ a *polynomial on* $\mathcal{V}$ if f is a polynomial over $\mathbb{K}^n$; the *degree* of $\overline{f}$ is that of f.

The natural projection

$$\pi : \mathbb{K}_{m-1}[x_1,\ldots,x_n] \mapsto \mathbb{K}_{m-1}[x_1,\ldots,x_n]/I(\mathcal{V})$$

induces an exact sequence

$$0 \to \ker(\pi) \to \mathbb{K}_{m-1}[x_1,\ldots,x_n] \to \mathbb{K}_{m-1}[x_1,\ldots,x_n]/I(\mathcal{V})$$

where $\mathbb{K}_{m-1}[x_1,\ldots,x_n]$ denotes the set of polynomials of degree $\leqslant m-1$. Hence, we have

$$\dim(\mathbb{K}_{m-1}[x_1,\ldots,x_n]/I(\mathcal{V})) = \dim(\mathbb{K}_{m-1}[x_1,\ldots,x_n]) - \dim(I(\mathcal{V}) \cap \mathbb{K}_{m-1}[x_1,\ldots,x_n]).$$

We denote this dimension by $d(m-1)$.

Example 2 (Llibre and Zhang [16])**.** If $\mathcal{V}$ is a smooth (or regular) hypersurface defined by a polynomial of degree d, then

$$d(m-1) = \dim(\mathbb{K}_{m-1}[x_1,\ldots,x_n]/I(\mathcal{V})) = \binom{n+m-1}{n} - \binom{n+m-1-d}{n}.$$

By definition a *polynomial vector field on* $\mathcal{V}$ is a vector field of the form

$$\mathbf{X} = \overline{P}_1 \frac{\partial}{\partial x_1} + \overline{P}_2 \frac{\partial}{\partial x_2} + \cdots + \overline{P}_n \frac{\partial}{\partial x_n} \qquad \text{on} \quad \mathcal{V}$$

with $P_i \in \mathbb{K}[x_1,\ldots,x_n]$, $\overline{P}_i \in \mathbb{K}[x_1,\ldots,x_n]/I(\mathcal{V})$ and $(\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n)$ belong to the tangent bundle of $\mathcal{V}$. We state that the polynomial vector field $\mathbf{X}$ on $\mathcal{V}$ has *degree* m if $\max\{\deg \overline{P}_i, i = 1,\ldots,n\} = m$.

For $\overline{f} \in \mathbb{K}[x_1,\ldots,x_n]/I(\mathcal{V})$, we state that $\{f = 0\} \cap \mathcal{V}$ is an *invariant algebraic variety* of the vector field $\mathbf{X}$ on $\mathcal{V}$ (or simply $\overline{f}$ is an invariant algebraic variety of $\mathbf{X}$ on $\mathcal{V}$) if the following statements hold:

(i) There exists a $\overline{k} \in \mathbb{K}[x_1,\ldots,x_n]/I(\mathcal{V})$ such that

$$\sum_{i=1}^{n} P_i \frac{\partial f}{\partial x_i} = kf \qquad \text{on} \quad \mathcal{V}.$$

The polynomial $\overline{k}$ on $\mathcal{V}$ is called the cofactor of $\overline{f}$ on $\mathcal{V}$.

(ii) The hypersurface $f = 0$ and the variety $\mathcal{V}$ have transversal intersection, i.e. for each $p \in \{f = 0\} \cap \mathcal{V}$, $\frac{\partial f}{\partial x}(p) \otimes \frac{\partial f_\alpha}{\partial x}(p) \neq 0, \alpha = 1,\ldots,k$, where $\otimes$ denotes the outer product of two vectors.

We note that for the polynomial vector field $\mathbf{X}$ on $\mathcal{V}$ of degree m, the cofactor $\overline{k}$ has degree at most $m-1$.

An *exponential factor* $\overline{F}$ of the vector field $\mathbf{X}$ on $\mathcal{V}$ is an exponential function of the form $\mathrm{e}^{\overline{g}/\overline{h}}$ with $\overline{g}, \overline{h} \in \mathbb{K}[x_1,\ldots,x_n]/I(\mathcal{V})$, $(f, g) = 1$ and satisfying $\mathbf{X}F = KF$ on $\mathcal{V}$ for some $\overline{K} \in \mathbb{K}_{m-1}[x_1,\ldots,x_n]/I(\mathcal{V})$. The polynomial $\overline{K}$ on $\mathcal{V}$ has degree at most $m-1$, and is called an exponential factor of $\overline{F}$ on $\mathcal{V}$.

The *singular point* of $\mathbf{X}$ on $\mathcal{V}$ is a point $x \in \mathcal{V}$ such that $P_i(x) = 0$ for $i = 1, \ldots, n$. It is well known that for each point $p = (p_1, \ldots, p_n) \in \mathbb{K}^n$, $m_p = I(x_1 - p_1, \ldots, x_n - p_n) \subset \mathbb{K}[x_1, \ldots, x_n]$ is a maximal ideal. We state that the r points $p^{(1)}, \ldots, p^{(r)}$ on $\mathcal{V}$ are *independent* if

$$\dim\left(\left(\bigcap_{i=1}^{r} m_{p^{(i)}}\right)\bigcap(\mathbb{K}_{m-1}[x_1, \ldots, x_n]/I(\mathcal{V}))\right) = d(m-1) - r.$$

Let $\mathcal{U} \subset \mathbb{K}^n$ be an open set. We state that a function $H(x,t) : \mathcal{U} \times \mathbb{K} \to \mathbb{K}$ is an *invariant* of the vector field $\mathbf{X}$ on $\mathcal{V} \cap \mathcal{U}$ if $H(x(t), t) \equiv$ constant for all values of t for which the solution $x(t)$ of $\mathbf{X}$ is defined on $\mathcal{V} \cap \mathcal{U}$. If an invariant H is independent of t, it is called a *first integral*. If a first integral H is a ratio of two analytic functions, it is called a *generalized rational first integral* (see [12]).

The following result provides an extension of the Darboux theory of integrability to polynomial vector fields on algebraic varieties.

Theorem 3. *Let $\mathcal{V}$ be an irreducible smooth algebraic variety in $\mathbb{K}^n$. Assume that the polynomial vector field $\mathbf{X}$ on $\mathcal{V}$ of degree m has μ invariant algebraic varieties $\overline{f}_i$ with cofactors k_i for $i = 1, \ldots, \mu$; ν exponential factors $\overline{F}_j = \mathrm{e}^{\overline{g}_j/\overline{h}_j}$ with cofactors K_j for $j = 1, \ldots, \nu$; and r independent singular points x_s of $\mathbf{X}$ on $\mathcal{V}$ such that $\overline{f}_i(x_s) \neq 0$ for $i = 1, \ldots, \mu$ and $s = 1, \ldots, r$. The following statements hold:*

(a) For $k < n$ we have

(a1) If there exist $\alpha_i, \beta_i \in \mathbb{K}$ not all zero such that

$$\sum_{i=1}^{\mu} \alpha_i k_i + \sum_{j=1}^{\nu} \beta_j K_j \in I(\mathcal{V}) \tag{2}$$

then the following function (multi-valued) of the Darbouxian type

$$\prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j} \qquad on \quad \mathcal{V} \tag{3}$$

is a first integral of the vector field $\mathbf{X}$ on $\mathcal{V}$.

(a2) If $\mu + \nu + r \geqslant d(m-1) + 1$, then there exist $\alpha_i, \beta_j \in \mathbb{K}$ not all zero such that (2) holds.

(a3) If there exist $\alpha_i, \beta_j \in \mathbb{K}$ not all zero such that $\sum_{i=1}^{\mu} \alpha_i k_i + \sum_{j=1}^{\nu} \beta_j K_j + \sigma \in I(\mathcal{V})$ for some $\sigma \in \mathbb{K}\backslash\{0\}$, then the following function (multi-valued)

$$\prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j} \mathrm{e}^{\sigma t}$$

is an invariant of $\mathbf{X}$ on $\mathcal{V}$.

(b) For $k = n-2$ and $\mathbb{K} = \mathbb{C}$, if $\mu + \nu + r \geqslant d(m-1) + 2$, the vector field $\mathbf{X}$ on $\mathcal{V}$ has a generalized rational first integral, and consequently all trajectories of the vector field are contained in invariant analytic curves.

We now give some examples to show the applications of theorem 3 to dynamical models.

Example 3. Under the conditions of example 1, the Lotka–Volterra system is Darbouxian integrable. Because we have the five cofactors satisfying $k_1 + k_2 + k_3 + k_4 - 2k_5 = 0$, hence

$$H(x, y) = f_1 f_2 f_3 f_4 f_5^{-2}$$

is a rational first integral of the system. Consequently, all the orbits of the system are given by the curves $\mu f_1 f_2 f_3 f_4 - \nu f_5^2 = 0$ with μ and ν arbitrary real constants and $\mu^2 + \nu^2 \neq 0$.

Example 4. For the Lorenz system [17]

$$\dot{x} = s(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = -bz + xy$$

with s, r, b real constants, in [21] we proved that if $s \neq 0$, it has a Darbouxian first integral if and only if $b = 1$, $s = \frac{1}{2}$ and $r = 0$. The first integral is $(y^2 + z^2)/(x^2 - z)^2$. Hence, in this case the Lorenz system has no chaotic phenomena.

Example 5. For the Rössler system [18]

$$\dot{x} = -(y + z) \qquad \dot{y} = x + ay \qquad \dot{z} = b - cz + xz$$

with a, b, c real constants, we proved [22] that it has a Darbouxian first integral if and only if $a = b = c = 0$. If it is the case, the system has a Darbouxian first integral $H_1 = z\,\mathrm{e}^{-y}$ and a polynomial first integral $H_2 = x^2 + y^2 + 2z$. Consequently, it is completely integrable. The orbits of the system are given by the curves $\{z\,\mathrm{e}^{-y} = c_1\} \cap \{x^2 + y^2 + 2z = c_2\}$, where c_1 and c_2 are arbitrary real constants.

2. The proof of theorem 1

We first introduce the following definition. The rth *extactic hypersurface* of $\mathbf{X}$ in $\mathbb{K}^n$ is defined by the equation

$$\mathcal{E}_r(\mathbf{X}) = \begin{vmatrix} v_1 & v_2 & \dots & v_l \\ \mathbf{X}(v_1) & \mathbf{X}(v_2) & \dots & \mathbf{X}(v_l) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}^{l-1}(v_1) & \mathbf{X}^{l-1}(v_2) & \dots & \mathbf{X}^{l-1}(v_l) \end{vmatrix} = 0$$

where $(v_1, \dots, v_l)$ is a basis of the $\mathbb{K}$-vector subspace $\mathbb{K}_r[x_1, \dots, x_n]$ of $\mathbb{K}[x_1, \dots, x_n]$, and $\mathbf{X}^j(v_i) = \mathbf{X}(\mathbf{X}^{j-1}(v_i))$. Here, we use the notation $\mathbf{X}^0(v_i) = v_i$. It is well known that $\dim \mathbb{K}_r[x_1, \dots, x_n] = \binom{n+r}{r}$. So, we have $l = \binom{n+r}{r}$. This definition is similar to that of [15], where the authors defined the extactic curves for a planar polynomial vector field.

We remark that the definition of the rth extactic hypersurface of the vector field $\mathbf{X}$ is independent of the chosen basis of $\mathbb{K}_r[x_1, \dots, x_n]$. The following result gives a relationship between the rth extactic hypersurface and the invariant algebraic hypersurfaces.

Lemma 4. *Every algebraic hypersurface of degree $\leqslant r$ invariant by the vector field $\mathbf{X}$ is a factor of $\mathcal{E}_r(\mathbf{X})$.*

Proof. Let f be an invariant algebraic hypersurface of degree $s \leqslant r$ for the vector field $\mathbf{X}$, and let the corresponding cofactor be k_f. Since $\mathcal{E}_r(\mathbf{X})$ does not dependent on the choice of the basis for $\mathbb{K}_r[x_1, \dots, x_n]$, we can take $v_i = f$ for some $1 \leqslant i \leqslant l = \binom{n+r}{r}$. Then for $\mathbf{X}(f) = k_f f$ we have

$$\begin{aligned} \mathbf{X}^2(f) &= \left(k_f^2 + \mathbf{X}(k_f)\right) f \\ \dots \quad & \quad \dots \\ \mathbf{X}^i(f) &= M_{i(m-1)} f \end{aligned}$$

where $M_{i(m-1)}$ is a polynomial of degree at most $i(m-1)$ in the variables $x_1, \dots, x_n$. So, the polynomial f divides $\mathcal{E}_r(\mathbf{X})$. This proves the lemma. □

Proof of theorem 1. We consider the 1st extactic hypersurface $\mathcal{E}_1(\mathbf{X})$. Choose $v_1 = 1, v_2 = x_1, \ldots, v_l = x_n$ as a basis of $\mathbb{K}_1[x_1, \ldots, x_n]$, where $l = n + 1$. Then $\mathbf{X}^j(v_1) = 0$ for $j = 1, \ldots, n$, and $\mathbf{X}(x_i) = P_i$ for $i = 1, \ldots, n$. Hence, we have

$$\mathcal{E}_1(\mathbf{X}) = \begin{vmatrix} P_1 & P_2 & \ldots & P_n \\ \mathbf{X}(P_1) & \mathbf{X}(P_2) & \ldots & \mathbf{X}(P_n) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{X}^{n-1}(P_1) & \mathbf{X}^{n-1}(P_2) & \ldots & \mathbf{X}^{n-1}(P_n) \end{vmatrix}.$$

Set $m = \max\{m_i, i = 1, \ldots, n\}$. Then an easy computation gives

$$\deg \mathbf{X}^j(P_i) \leqslant j(m-1) + m_i \qquad i = 1, \ldots, n \quad j = 1, \ldots, n-1. \tag{4}$$

From the definition of determinants, we get

$$\mathcal{E}_1(\mathbf{X}) = \sum (-1)^s \mathbf{X}^0\big(P_{i_1}\big)\mathbf{X}\big(P_{i_2}\big)\ldots\mathbf{X}^{n-1}\big(P_{i_n}\big)$$

where $i_1 i_2, \ldots, i_n$ is a permutation without repetition of $1, 2, \ldots, n$, the sum runs over all permutations without repetition of $1, 2, \ldots, n$, and s is the number of inverted sequences for the permutation $i_1 i_2, \ldots, i_n$ with respect to the standard sequence $1, 2, \ldots, n$.

Using inequality (4), we have

$$\begin{aligned} \deg\big(\mathbf{X}^0\big(P_{i_1}\big)\mathbf{X}\big(P_{i_2}\big)\ldots\mathbf{X}^{n-1}\big(P_{i_n}\big)\big) &\leqslant m_{i_1} + \big((m-1) + m_{i_2}\big) + \cdots + \big((n-1)(m-1) + m_{i_n}\big) \\ &= \sum_{i=1}^{n} m_i + (m-1)\binom{n}{2}. \end{aligned}$$

This implies that

$$\deg \mathcal{E}_1(\mathbf{X}) \leqslant \sum_{i=1}^{n} m_i + (m-1)\binom{n}{2}.$$

Combining lemma 4 and this last inequality, we have finished the proof of the theorem. □

We remark that since $\dim \mathbb{K}_1[x_1, \ldots, x_n] = n + 1$, the vector field $\mathbf{X}$ has at most $n + 1$ independent invariant hyperplanes.

3. The proof of theorem 2

Proof of statement (a). We first provide the following results without proof, because it is easy to check.

Proposition 5. *For the polynomial vector field $\mathbf{X}$ of degree $\mathbf{m} = (m_1, \ldots, m_n)$, the following statements hold:*

(i) If $P_n = 0$, the vector field $\mathbf{X}$ has a first integral $H = x_n$, and consequently it has infinitely many invariant hyperplanes.

(ii) If $m_{n-1} = m_n = 0$ and $P_{n-1}P_n \neq 0$, the vector field $\mathbf{X}$ has a first integral $H = P_n x_{n-1} - P_{n-1} x_n$, and consequently it has infinitely many invariant hyperplanes.

From this proposition, statement (a) follows. □

Proof of statement (b). Consider the following linear system

$$\dot{x}_i = a_{i0} + a_{i1}x_1 + \cdots + a_{in}x_n = P_i \qquad i = 1, \ldots, n. \tag{5}$$

Let $f = b_0 + b_1x_1 + \cdots + b_nx_n = 0$ be an invariant hyperplane with a constant cofactor k. From the definition of invariant hyperplane, i.e. the equation

$$\sum_{i=1}^{n} P_i \frac{\partial f}{\partial x} = kf \tag{6}$$

we get

$$\begin{pmatrix} -k & a_{10} & a_{20} & \dots & a_{n0} \\ 0 & a_{11}-k & a_{21} & \dots & a_{n1} \\ 0 & a_{12} & a_{22}-k & \dots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{1n} & a_{2n} & \dots & a_{nn}-k \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = 0. \tag{7}$$

From proposition 5, in what follows we consider only two cases: $\mathbf{m} = (1, \dots, 1)$ and $\mathbf{m} = (0, 1, \dots, 1)$ with $P_1 \neq 0$.

Case 1. $m_1 = \cdots = m_n = 1$. By selecting suitable a_{ij} for $i, j = 1, \dots, n$, the following matrix

$$A = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

has exactly n different eigenvalues not equal to zero, denoted by $k_1, \dots, k_n$. Corresponding to each eigenvalue k_i, the matrix A has a unique independent eigenvector, denoted by $\left(b_1^{(i)}, \dots, b_n^{(i)}\right)$. Using k_i instead of k in (7), since $k_i \neq 0$, we get that the system of linear equations (7) has a unique non-zero independent solution $\left(b_0^{(i)}, b_1^{(i)}, \dots, b_n^{(i)}\right)$. This implies that there exist systems (5) of degree $\mathbf{m} = (1, \dots, 1)$ with exactly n invariant hyperplanes. This proves statement (b) in the case $\mathbf{m} = (1, \dots, 1)$.

Case 2. $\mathbf{m} = (0, 1, \dots, 1)$ and $P_1 = a_{10} \neq 0$. Consider the algebraic system (7) with $a_{11} = a_{12} = \cdots = a_{1n} = 0$. It is clear that there exist matrices

$$B = \begin{pmatrix} a_{22} & \dots & a_{n2} \\ \vdots & \ddots & \vdots \\ a_{2n} & \dots & a_{nn} \end{pmatrix}$$

with exactly $n-1$ different eigenvalues not equal to zero, denoted by $k_2, \dots, k_n$. Corresponding to each k_i, the linear equation $(B - k_iE)\overline{b}_i = 0$ has a unique non-zero independent solution $\overline{b}_i = \left(b_2^{(i)}, \dots, b_n^{(i)}\right)$, where E is the unit matrix of order $n-1$. Working in a way similar to the proof of case 1, we get that the algebraic equation (7) has $n-1$ different solutions. Therefore, there exist systems (5) of degree $\mathbf{m} = (0, 1, \dots, 1)$ such that they have exactly $n-1$ invariant hyperplanes. Hence, we have proved statement (b). □

Proof of statement (c). From the proof of case 1 of statement (b), we get that the existence of n invariant hyperplanes for the vector field $\mathbf{X}$ depends on the number of eigenvectors of the matrix A. It is well known that the subspace of the square matrices of order n with non-zero eigenvalues and n different eigenvectors is open and dense in the space of all square matrices of order n. This implies that statement (c) holds in the case $m_1 = \cdots = m_n = 1$.

Combining equation (7) and the matrix B, and working in a way similar to the proof of the last paragraph, we can prove statement (c) in the case $m_1 = 0$ and $m_2 = \cdots = m_n = 1$. This completes the proof of statement (c). So, we have finished the proof of the theorem. □

4. The proof of theorem 3

From the assumptions and the definitions of invariant algebraic varieties and exponential factors, it follows that on the variety $\mathcal{V}$, $\mathbf{X} f_i = k_i f_i$ for $i = 1, \ldots, \mu$, and $\mathbf{X} F_j = K_j F_j$ for $j = 1, \ldots, \nu$.

Proof of statement (a1). Since there exist $\alpha_i, \beta_j \in \mathbb{K}$ not all zero such that (2) holds, i.e. $\sum_{i=1}^{\mu} \alpha_i k_i + \sum_{j=1}^{\nu} \beta_j K_j = 0$ on $\mathcal{V}$, we have

$$\mathbf{X}\left(\prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j}\right) = \prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j} \left(\sum_{i=1}^{\mu} \alpha_i k_i + \sum_{j=1}^{\nu} \beta_j K_j\right) = 0 \qquad \text{on} \quad \mathcal{V}.$$

This shows that (3) is a first integral of the vector field $\mathbf{X}$ on $\mathcal{V}$. □

Proof of statement (a2). From the assumptions, we have

$$\sum_{i=1}^{n} \overline{P}_i(x) \frac{\partial \overline{f}_j}{\partial x_i} = \overline{k}_j(x) \overline{f}_j(x) \qquad j = 1, \ldots, \mu$$

$$\sum_{i=1}^{n} \overline{P}_i(x) \frac{\partial \overline{f}_j}{\partial x_i} = \overline{K}_j(x) \overline{F}_j(x) \qquad j = 1, \ldots, \nu.$$

Since x_s for $s = 1, \ldots, r$ are singular points of the vector field $\mathbf{X}$ on $\mathcal{V}$ and $\overline{f}_i(x_s)\overline{F}_j(x_s) \neq 0$ for $i = 1, \ldots, \mu$, $j = 1, \ldots, \nu$ and $s = 1, \ldots, r$, we get that $\overline{k}_i(x_s) = 0$ for $i = 1, \ldots, \mu$ and $s = 1, \ldots, r$, and $\overline{K}_j(x_s) = 0$ for $j = 1, \ldots, \nu$ and $s = 1, \ldots, r$. This means that $k_i, K_j \in \bigcap_{s=1}^{r} m_{x_s}$, and so $\overline{k}_i, \overline{K}_j \in \left(\bigcap_{s=1}^{r} m_{x_s}\right) \bigcap (\mathbb{K}_{m-1}[x_1, \ldots, x_n]/I(\mathcal{V}))$. Since the linear space $\left(\bigcap_{s=1}^{r} m_{x_s}\right) \bigcap (\mathbb{K}_{m-1}[x_1, \ldots, x_n]/I(\mathcal{V}))$ has the dimension $d(m-1) - r$, it follows from the assumptions that $\overline{k}_1, \ldots, \overline{k}_\mu; \overline{K}_1, \ldots, \overline{K}_\nu$ are linearly dependent in the space $\left(\bigcap_{s=1}^{r} m_{x_s}\right) \bigcap (\mathbb{K}_{m-1}[x_1, \ldots, x_n]/I(\mathcal{V}))$. Hence, there exist $\alpha_i, \beta_j \in \mathbb{K}$ not all zero with $i = 1, \ldots, \mu$; $j = 1, \ldots, \nu$ such that

$$\sum_{i=1}^{\mu} \alpha_i k_i(x) + \sum_{j=1}^{\nu} \beta_j K_j(x) \in I(\mathcal{V}) \cap \mathbb{K}_{m-1}[x_1, \ldots, x_n].$$

This proves the statement. □

Proof of statement (a3). The straightforward calculations give that on the variety $\mathcal{V}$

$$\mathbf{X}\left(\prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j} \mathrm{e}^{\sigma t}\right) = \prod_{i=1}^{\mu} f_i^{\alpha_i} \prod_{j=1}^{\nu} F_j^{\beta_j} \mathrm{e}^{\sigma t} \left(\sum_{i=1}^{\mu} \alpha_i k_i + \sum_{j=1}^{\nu} \beta_j k_j + \sigma\right) = 0.$$

This implies statement (a3). □

Proof of statement (b). Applying statement (a2) to two subsets with $\mu + \nu + r - 1$ elements of the set formed by all the invariant algebraic varieties and the exponential factors, and after some linear algebra and relabelling (if necessary), we get two linear dependences on $\mathcal{V}$ among the corresponding cofactors

$$L_1 + \alpha_3 L_3 + \alpha_4 L_4 + \cdots + \alpha_{\mu+\nu} L_{\mu+\nu} \in I(\mathcal{V})$$
$$L_2 + \beta_3 L_3 + \beta_4 L_4 + \cdots + \beta_{\mu+\nu} L_{\mu+\nu} \in I(\mathcal{V})$$

where L_s are the cofactors k_i and K_j, and $\alpha_i, \beta_j \in \mathbb{K}$. Correspondingly, M_s represents the invariant algebraic variety or the exponential factor associated with the cofactor L_s. Then from statement (a1) the vector field $\mathbf{X}$ on $\mathcal{V}$ has two first integrals (multi-valued) of the Darbouxian type,

$$M_1 M_3^{\alpha_3} \ldots M_{\mu+\nu}^{\alpha_{\mu+\nu}} \qquad M_2 M_3^{\beta_3} \ldots M_{\mu+\nu}^{\beta_{\mu+\nu}}.$$

Taking logarithms to the above two first integrals, we obtain two other first integrals

$$\begin{aligned} H_1 &= \ln M_1 + \alpha_3 \ln M_3 + \cdots + \alpha_{\mu+\nu} \ln M_{\mu+\nu} \\ H_2 &= \ln M_2 + \beta_3 \ln M_3 + \cdots + \beta_{\mu+\nu} \ln M_{\mu+\nu} \end{aligned} \tag{8}$$

of the vector field $\mathbf{X}$ on $\mathcal{V}$.

An algebraic variety in $\mathbb{C}^n$ is clearly an analytic subvariety. Under the assumptions of the theorem, the variety $\mathcal{V}$ is a complex manifold of dimension 2. So, there exist an open cover $\{U_\alpha\}$ of $\mathcal{V}$ and coordinate maps $\phi_\alpha : U_\alpha \to \mathbb{C}^2$ such that $\phi_\alpha \phi_\beta^{-1}$ is holomorphic on $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^2$ for all α, β. Moreover, from the assumptions on the variety $\mathcal{V}$ and using the implicit function theorem we may select ϕ_α such that $\phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$.

The vector field $\mathbf{X}$ restricted to U_α induces an analytic vector field $\mathbf{X}_\alpha$ in $\mathbb{C}^2$. It has two first integrals

$$\begin{aligned} H_1^\alpha &= \ln M_1^\alpha + \alpha_3 \ln M_3^\alpha + \cdots + \alpha_{\mu+\nu} \ln M_{\mu+\nu}^\alpha \\ H_2^\alpha &= \ln M_2^\alpha + \beta_3 \ln M_3^\alpha + \cdots + \beta_{\mu+\nu} \ln M_{\mu+\nu}^\alpha \end{aligned} \tag{9}$$

where M_i^α is an analytic function formed by M_i via the holomorphic coordinate transformation ϕ_α.

We write the analytic system in $\mathbb{C}^2$ associated with $\mathbf{X}_\alpha$ in the form

$$\dot{u} = P_\alpha(u, v) \qquad \dot{v} = Q_\alpha(u, v) \qquad (u, v) \in \mathbb{C}^2. \tag{10}$$

It has the first integrals H_1^α and H_2^α. Each first integral H_i^α provides an integrating factor R_i^α of system (10), which satisfies

$$P_\alpha R_i^\alpha = \frac{\partial H_i^\alpha}{\partial v} \qquad Q_\alpha R_i^\alpha = -\frac{\partial H_i^\alpha}{\partial u}.$$

So, we have

$$\frac{R_1^\alpha}{R_2^\alpha} = \frac{\frac{\partial H_1^\alpha}{\partial v}}{\frac{\partial H_2^\alpha}{\partial v}}. \tag{11}$$

This is a generalized rational first integral of (10). It follows from the fact that $\frac{\partial H_i^\alpha}{\partial v}$ for $i = 1, 2$ are generalized rational functions, and that if R_1 and R_2 are two integrating factors of a planar vector field, then R_1/R_2 is a first integral of the vector field (of course, $R_1/R_2 \not\equiv$ constant).

The composition of ϕ_α with R_1^α/R_2^α provides a generalized rational first integral of $\mathbf{X}$ on U_α, denoted by $H^\alpha = \frac{f^\alpha}{g^\alpha}$, where f^α and g^α are two analytic functions on U_α. Since we choose the ϕ_α such that $\phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$. Hence, we obtain from (8), (9) and (11) that $f^\alpha = f^\beta$ and $g^\alpha = g^\beta$ on $U_\alpha \cap U_\beta$. For $x \in \mathcal{V}$, set $f(x) = f^\alpha(x)$ and $g(x) = g^\alpha(x)$ if $x \in U_\alpha$. Then $f(x)$ and $g(x)$ are two globally defined analytic functions on $\mathcal{V}$, and so $H(x) = \frac{f(x)}{g(x)}$ provides a generalized rational first integral of the vector field $\mathbf{X}$ on $\mathcal{V}$. This proves the first part of the statement.

From the proof of the first part, we get a generalized rational first integral $H = \frac{f}{g}$, where f and g are two holomorphic functions on $\mathcal{V}$. For $\mathcal{V}$ dimension $n-2$, the vector field $\mathbf{X}$ on $\mathcal{V}$ is integrable; its integral curves are contained in the set $\{g = 0, \text{ or } f/g = c, c \in \mathbb{C}\}$. Obviously, the curves defined by $g = 0$ or $f - cg = 0$ are analytic. Hence, we have finished the proof of the theorem. □

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